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# Classical and Quantum Dynamics of Constrained Hamiltonian Systems

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# Preface

Since the pioneering work of Bergmann in 1949, it has been understood that there exists a direct connection between local symmetries of a Lagrangian and the existence of constraints contained in the Euler-Lagrange equations of motion. It was Dirac who subsequently discussed the Hamiltonian dynamics of constrained systems in a systematic way. Since then numerous papers, with applications to a wide range of problems, have been written on this subject. It is remarkable that even after more than half a century, this subject still raises questions to be answered.

This book is the result of lectures delivered by the authors at the University of Heidelberg. It is intended as an introduction to this interesting field of research, starting from the early work of Dirac up to more recent work, such as the Field-Antifield formalism of Batalin and Vilkovisky. Also included is a brief discussion of gauge anomalies within this formalism. Here we have restricted ourselves to the essentials, and have illustrated the ideas in terms of a simple example, the chiral Schwinger model.

Great emphasis is placed on discussing the subject of constrained Hamiltonian systems in a transparent way without becoming too technical. Proofs, as well as examples, are discussed in great detail, at the risk of being sometimes pedantic. Each example has been chosen to illustrate some important point. The book should thus hopefully provide a good basis for master degree and Ph.D students.

The book is divided into two parts: part one, involving chapters 1 through 8, deals with the classical dynamics of constrained Hamiltonian systems, while the second part is devoted to their quantization. Here it is assumed that the reader is familiar with the Feynman path integral approach to quantization.

$Q_B$ : generator of BRST transformations:  $\delta_B \mathcal{F} = \epsilon \{Q_B, \mathcal{F}\} = \{\mathcal{F}, Q_B\} \epsilon$

$s$ : operator inducing "left" BRST transformations:  $s\mathcal{F} = \{Q_B, \mathcal{F}\}$

$\bar{s}$ : operator inducing "right" BRST transformations  $\bar{s}\mathcal{F} = \{\mathcal{F}, Q_B\}$

$\delta$ : operator inducing "right" BRST transformations on Lagrangian level

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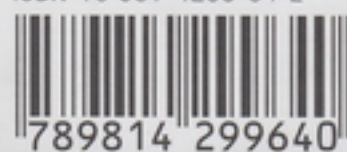
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# Classical and Quantum Dynamics of Constrained Hamiltonian Systems

This book is an introduction to the field of constrained Hamiltonian systems and their quantization, a topic which is of central interest to theoretical physicists who wish to obtain a deeper understanding of the quantization of gauge theories, such as describing the fundamental interactions in nature. Beginning with the early work of Dirac, the book covers the main developments in the field up to more recent topics, such as the field–antifield formalism of Batalin and Vilkovisky, including a short discussion of how gauge anomalies may be incorporated into this formalism. All topics are well illustrated with examples emphasizing points of central interest. The book should enable graduate students to follow the literature on this subject without much problems, and to perform research in this field.

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